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CONVERSES OF LOEWNER-HEINZ INEQUALITY VIA OPERATOR MEANS

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ABSTRACT. Let $f(t)$ be an operator monotone function. Then $A \leq B$ implies $f(A) \leq f(B)$, moreover $f(A) \leq f(B)$ implies $f(A)^{-1}\sharp f(B) \leq I$. But the converse implications are not true. We will show that if $(I+\frac{k}{n}B)^{-1}\sharp(I+\frac{k}{n}A) \leq I$ for all $0 < k \leq n$, then $A \leq B$. Moreover, we extend it to multi-variable matrices means.

1. INTRODUCTION

In what follows, $\mathcal{H}$ means a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$, and an operator means a bounded linear operator on $\mathcal{H}$. An operator A is said to be positive (denoted by $A \geq 0$) if and only if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$, and $A \leq B$ means $B - A$ is positive. Moreover, an operator A is said to be positive definite (denoted by $A > 0$) if A is positive and invertible.

A real continuous function $f(t)$ defined on a real interval I is said to be *operator monotone*, provided $A \leq B$ implies $f(A) \leq f(B)$ for any two bounded self-adjoint operators A and B whose spectra are in I. Typical examples of operator monotone functions are t^a for $0 < a < 1$ and $\log t$. Lowener-Heinz inequality means that $A^a \leq B^a$ for $0 < a < 1$ if $A \leq B$ for positive operators A and B. A continuous function f defined on I is called an *operator convex function* on I if $f(sA+(1-s)B) \leq sf(A)+(1-s)f(B)$ for every $0 < s < 1$ and for every pair of bounded self-adjoint operators A and B whose spectra are both in I. An *operator concave function* is likewise defined. If $I = (0,\infty)$, then $f(t)$ is operator monotone on I if and only if $f(t)$ is operator concave and $f(\infty) > -\infty$ ([14], cf.[5]). This implies that every operator monotone function on $(0,\infty)$ is operator concave. Then the associated operator mean $A\sigma B$ is defined and represented as

$$A\sigma B = A^{\frac{1}{2}} f(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}} \tag{1.1}$$

if A is invertible [7]. σ is said to be symmetric if $A\sigma B = B\sigma A$ for every A, B. σ is symmetric if and only if $f(t) = tf(1/t)$. When $f(t) = t^a$ $(0 < a < 1)$, the associated mean is denoted by $A\sharp_a B$ and called weighted geometric mean. In particular, the case of $a = \frac{1}{2}$ is the usual geometric mean and simply denoted by $A\sharp B$. The arithmetic mean ∇ and the harmonic mean $!$ are naturally defined. It is well-known that $A!B \leq$

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$A\sharp B \le A\nabla B$ for every $A, B \ge 0$; of course these are symmetric. It is well-known that $0 < A \le B$ implies that $B^{-1}\sharp A \le A^{-1}\sharp A = I$, but the converse does not hold.

In the recent years, geometric means of n-matrices are studied by many authors. Let $\mathbb{P}_m$ be the set of all m–by–m positive definite matrices. Define $\omega = (w_1, ..., w_n)$ be a probability vector, i.e., $w_i > 0$ for $i = 1, ..., n$ and $\sum_{i=1}^n w_i = 1$. Let Δ_n be the set of all probability vectors. For $\omega = (w_1, ..., w_n) \in \Delta_n$, the *Karcher mean* $\Lambda(\omega; A_1, ..., A_n)$ of $A_1, ..., A_n \in \mathbb{P}_m$ is characterized as the unique positive definite solution of the matrix equation [12]

$$\sum_{i=1}^{n} w_i \log(X^{\frac{-1}{2}} A_i X^{\frac{-1}{2}}) = 0.$$

If $\omega = (\frac{1}{n}, ..., \frac{1}{n}) \in \Delta_n$, then the Karcher mean is simply written by $\Lambda(A_1, ..., A_n)$. In the two matrices case, $A, B \in \mathbb{P}_m$, the Karcher mean coincides with the weighted geometric mean. We note that the above matrix equation is called the Karcher equation [6]. The Katcher mean inherits many properties of geometric means (see [2, 12, 9, 3]). For instance, $\sum_{i=1}^n w_i A_i \le I$ implies $\Lambda(\omega; A_1, ..., A_n) \le I$ for $\omega = (w_1, ..., w_n) \in \Delta_n$ in [11, 16].

Related to the Karcher mean, the power mean is also discussed in [10]. The power mean of n-matrices is inspired from the power mean of positive numbers. For $t \in [-1, 1]\backslash\{0\}$ and $\omega = (w_1, ..., w_n) \in \Delta_n$, the power mean $P_t(\omega; A_1, ..., A_n)$ of $A_1, ..., A_n \in \mathbb{P}_m$ is defined as the unique positive definite solution of the matrix equation

$$\sum_{i=1}^{n} w_i (X \sharp_t A_i) = X, \tag{1.2}$$

where if $t \in [-1, 0)$, $X\sharp_t A_i$ means $X^{\frac{1}{2}}(X^{\frac{-1}{2}} A_i X^{\frac{-1}{2}})^t X^{\frac{1}{2}}$, but it is not an operator mean. If $\omega = (\frac{1}{n}, ..., \frac{1}{n}) \in \Delta_n$, then the power mean is simply written by $P_t(A_1, ..., A_n)$. It is shown in [10] that the power mean of two matrices, $A, B \in \mathbb{P}_m$, coincides with

$$P_t(1 - w, w; A, B) = A^{\frac{1}{2}} \left((1 - w)I + w(A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^t\right)^{\frac{1}{t}} A^{\frac{1}{2}}.$$

The power mean interpolates among the arithmetic, Karcher (geometric) and harmonic means. More precisely, the Karcher mean can be considered as the limit point of the power mean as $t \to 0$, it is the same situation to the number case.

One of the author has obtained the following result:

Theorem A ([15]). *Let $f(t)$ be a non-constant operator monotone function with $f(1) > 0$. Then there exists $\{t_n\}_{n=1}^{\infty} \subset \mathbb{R}$ so that $t_n \downarrow 0$;*

$$A \le B \iff f(a + t_n A) \le f(a + t_n B).$$

Here we observe that for positive invertible operators A and B, the following implications hold:

$$A \le B \implies A^\alpha \le B^\alpha \ \alpha \in (0,1) \implies \log A \le \log B \implies A\sharp B^{-1} \le I. \tag{1.3}$$

Hence, we have the following question:

Question. *Let $f(t)$ be a non-constant operator monotone function with $f(1) > 0$. Then does there exist $\{t_n\}_{n=1}^{\infty} \subset \mathbb{R}$ so that $t_n \downarrow 0$;*

$$A \leq B \iff f(a + t_n A) \sharp f(a + t_n B)^{-1} \leq I?$$

The aim of this paper is to give an answer for the above question, and investigate the converse of Loewner-Heinz inequality in the view point of operator mean. It is organized as follows: In Section 2, we shall give an answer for the question, firstly. Then we shall show that if $f(\lambda A + I) \sigma f(\lambda B + I) \leq I$ for all operator mean satisfying $! \leq \sharp \leq \nabla$ and all sufficiently small $\lambda \geq 0$ if and only if $A \leq B$. In Section 3, we will extend the results obtained in Section 2 in the case of the power means and the Karcher mean.

2. Operator inequality and Operator mean

We begin by recalling a few results which we will need later. If $A \sharp B \leq I$, then $A^p \sharp B^p \leq I$ for all $p \geq 1$ (we call it Ando-Hiai inequality [1]). Actually, $A^p \sharp B^p$ is decreasing for $p \geq 1$ if $A \sharp B \leq I$ (see Corollary 3.3 of [13]). The following well-known result for positive invertible operators is essential (see [4]):

$$\log A \leq \log B \iff B^{-p} \sharp A^p \leq I \quad \text{for all } p \geq 0. \tag{2.1}$$

In this paper we deal with a non-constant operator monotone function $f(t)$ defined on a neighborhood of $t = t_0$. However we assume $t_0 = 1$ for simplicity. In this case, for every bounded self-adjoint operator A the function $f(\lambda A + I)$ is well-defined for sufficiently small λ. We also note that $f'(1) > 0$.

At the beginning of this section we give an answer for the question introduced in the previous section:

Answer. *For positive invertible operators A and B,*

$$A \leq B \iff \left(I + \frac{k}{n} A\right) \sharp \left(I + \frac{k}{n} B\right)^{-1} \leq I.$$

for all $0 < k \leq n$.

To prove this, we shall use so-called Ando-Hiai inequality: For positive invertible operators A and B,

$$A \sharp_a B \leq I \implies A^p \sharp_a B^p \leq I$$

holds for all $p \geq 1$.

Proof. ($\Longrightarrow$): Obvious by (1.3). ($\Longleftarrow$): By Ando-Hiai inequality,

$$\left(I + \frac{k}{n} A\right)^n \sharp \left(I + \frac{k}{n} B\right)^{-n} \leq I \quad \text{for all } n \geq 1.$$

Letting $n \to \infty$, we have

$$e^{kA} \sharp e^{-kB} \leq I \quad \text{for all } k > 0.$$

It is equivalent to $\log e^A \leq \log e^B$, i.e., $A \leq B$. □

We have the following results by investigating the above discussion.

Theorem 1. *Let $f(t)$ be an operator monotone function on $(0,\infty)$ with $f(1)=1$, and let A and B be bounded self-adjoint operators. Let σ be an operator mean satisfying $! \le \sigma \le \nabla$. Then $A \le B$ if and only if $f(\lambda A+I)\sigma f(-\lambda B+I) \le I$ for all sufficiently small $\lambda \ge 0$.*

To prove Theorem 1, we will use the following well-known lemma.

Lemma 2. *For positive invertible operators $A_1, ..., A_n$ and $\omega = (w_1, ..., w_n) \in \Delta_n$,*

$$\lim_{p\searrow 0}\left(\sum_{i=1}^n w_i A_i^p\right)^{\frac{1}{p}} = \exp\left(\sum_{i=1}^n w_i \log A_i\right),$$

uniformly, i.e., $\|\left(\sum_{i=1}^n w_i A_i^p\right)^{\frac{1}{p}} - \exp\left(\sum_{i=1}^n w_i \log A_i\right)\| \to 0$ *as* $p \searrow 0$.

Proof of Theorem 1. Assume $A \le B$. Since $\frac{(\lambda A+I)+(-\lambda B+I)}{2} \le I$ holds for every positive number λ and $f(1)=1$, we have

$$\begin{aligned} I &\ge f(\frac{(\lambda A+I)+(-\lambda B+I)}{2}) \ge \frac{f(\lambda A+I)+f(-\lambda B+I)}{2} \\ &= f(\lambda A+I)\nabla f(-\lambda B+I) \ge f(\lambda A+I)\sigma f(-\lambda B+I), \end{aligned}$$

where the second inequality is due to the operator concavity of f. Assume conversely $f(\lambda A+I)\sigma f(-\lambda B+I) \le I$. By the assumption we have $f(\lambda A+I)!f(-\lambda B+I) \le I$. Since $t^{\frac{\lambda}{p}}$ is operator concave for $0<\lambda\le p$, we observe

$$\left(\frac{f(\lambda A+I)^{\frac{-p}{\lambda}}+f(-\lambda B+I)^{\frac{-p}{\lambda}}}{2}\right)^{\frac{-\lambda}{p}} \le \left(\frac{f(\lambda A+I)^{-1}+f(-\lambda B+I)^{-1}}{2}\right)^{-1} \le I,$$

and then

$$\left(\frac{f(\lambda A+I)^{\frac{-p}{\lambda}}+f(-\lambda B+I)^{\frac{-p}{\lambda}}}{2}\right)^{\frac{-1}{p}} \le I.$$

In virtue of

$$\lim_{\lambda\to 0}||f(\lambda A+I)^{1/\lambda} - \exp(f'(1)A)|| = 0, \tag{2.2}$$

we obtain

$$\left(\frac{e^{-f'(1)pA}+e^{f'(1)pB}}{2}\right)^{\frac{-1}{p}} \le I \quad \text{as } \lambda\to 0.$$

Letting $p\to 0$, by Lemma 2, it yields $\exp(\frac{f'(1)}{2}(A-B)) \le I$. This implies $A \le B$. $\square$

We remark that a symmetric operator mean σ, that is $A\sigma B = B\sigma A$ for every A and B, satisfies $! \le \sigma \le \nabla$.

Theorem 3. *Let $f(t)$ be a non-constant operator monotone function on $(0,\infty)$ with $f(1)=1$, and let A and B be bounded self-adjoint operators. Then the following are equivalent:*

(i) $A \le B$,

(ii) $\|x\|^2 \le \|f(\lambda A+I)^{\frac{-1}{2}}x\|\|f(-\lambda B+I)^{\frac{-1}{2}}x\|$ *for all $x \in \mathcal{H}$ and all sufficiently small $\lambda \ge 0$,*

(iii) $\|x\|^2 \leq \|e^{-pA}x\| \|e^{pB}x\|$ *for all* $x \in \mathcal{H}$ *and all* $p \geq 0$.

To prove Theorem 3, we need the following lemma:

Lemma 4. *Let* $S_1, ..., S_n$ *be operators on* $\mathcal{H}$. *Then the following are mutually equivalent:*

(i) $I \leq \frac{1}{n}\sum_{i=1}^{n} t_i S_i^* S_i$ *for all* $t_1, ..., t_n > 0$ *with* $\prod_{i=1}^n t_i = 1$,

(ii) $\|x\|^n \leq \prod_{i=1}^{n} \|S_i x\|$ *for all* $x \in \mathcal{H}$.

Proof. Assume (i). Notice that each S_i is non-singular: indeed, if $S_i x = 0$ for a vector $x \in \mathcal{H}$, then there is a $\{t_i\}_{i=1}^n$ such that

$$\sum_{i=1}^{n} \frac{t_i}{n} \langle S_i^* S_i x, x\rangle < \langle x, x\rangle$$

and $\prod_{i=1}^n t_i = 1$. Since

$$\langle x, x\rangle \leq \sum_{i=1}^{n} \frac{t_i}{n} \langle S_i^* S_i x, x\rangle$$

for all $x \in \mathcal{H}$, by putting t_i as

$$t_i = \frac{\prod_{j=1}^n \langle S_j^* S_j x, x\rangle^{\frac{1}{n}}}{\langle S_i^* S_i x, x\rangle},$$

we have

$$\langle x, x\rangle \leq \sum_{i=1}^{n} \frac{t_i}{n} \langle S_i^* S_i x, x\rangle = \prod_{i=1}^{n} \|S_i x\|^{\frac{2}{n}}.$$

We consequently get (ii). Next assume (ii). For $t_1, ..., t_n > 0$ with $\prod_{i=1}^n t_i = 1$, we have

$$\|x\|^2 \leq \prod_{i=1}^{n} \|S_i x\|^{\frac{2}{n}} = \prod_{i=1}^{n} t_i^{\frac{1}{n}} \langle S_i^* S_i x, x\rangle^{\frac{1}{n}} \leq \sum_{i=1}^{n} \frac{t_i}{n} \langle S_i^* S_i x, x\rangle.$$

This yields (i). $\square$

Proof of Theorem 3. By Theorem 1, $A \leq B$ is equivalent to $f(\lambda A + I) \sharp f(-\lambda B + I) \leq I$ for all sufficiently small $\lambda \geq 0$. Then we have

$$\begin{aligned} I \geq f(\lambda A + I) \sharp f(-\lambda B + I) &= (t f(\lambda A + I)) \sharp \left(\frac{1}{t} f(-\lambda B + I)\right) \\ &\geq (t f(\lambda A + I))! \left(\frac{1}{t} f(-\lambda B + I)\right) \end{aligned}$$

for all $t > 0$, and obtain

$$I \leq \frac{\frac{1}{t} f(\lambda A + I)^{-1} + t f(-\lambda B + I)^{-1}}{2}$$

for all $t > 0$. By Lemma 4, we have (ii). Next we assume (ii). By Lemma 4

$$I \leq \frac{\frac{1}{t}f(\lambda A+I)^{-1}+tf(-\lambda B+I)^{-1}}{2}$$
$$\leq \left[\frac{\{\frac{1}{t}f(\lambda A+I)^{-1}\}^{\frac{p}{\lambda}}+\{tf(-\lambda B+I)^{-1}\}^{\frac{p}{\lambda}}}{2}\right]^{\frac{\lambda}{p}}$$

for all $0 < \lambda \leq p$ and all $t > 0$, where the last inequality follows from operator concavity of $t^{\frac{\lambda}{p}}$ for $\lambda/p \in [0,1]$. Then we have

$$I \leq \frac{(\frac{1}{t})^{\frac{p}{\lambda}}f(\lambda A+I)^{\frac{-p}{\lambda}}+t^{\frac{p}{\lambda}}f(-\lambda B+I)^{\frac{-p}{\lambda}}}{2}.$$

It is equivalent to

$$\|x\|^2 \leq \|f(\lambda A+I)^{\frac{-p}{2\lambda}}x\|\|f(-\lambda B+I)^{\frac{-p}{2\lambda}}x\|$$

for all $0 < \lambda \leq p$ and $x \in \mathcal{H}$ by Lemma 4. Letting $\lambda \to 0$, we have (iii) by (2.2) and replacing $\frac{pf'(1)}{2}$ into p. Lastly, we will prove (iii) $\longrightarrow$ (i). By Lemma 4, (iii) implies

$$I \leq \frac{e^{-2pA}+e^{2pB}}{2},$$

and then

$$I \leq \left(\frac{e^{-2pA}+e^{2pB}}{2}\right)^{\frac{1}{p}}$$

for all $p > 0$. By Lemma 2, we have

$$I \leq \exp\left(\frac{\log e^{-2A}+\log e^{2B}}{2}\right) = \exp(B-A).$$

This implies $A \leq B$. □

Corollary 5. *Let A and B be positive invertible operators. Then* $\log A \leq \log B$ *if and only if $\|x\|^2 \leq \|A^{-p}x\|\|B^px\|$ for all $p \geq 0$ and all $x \in \mathcal{H}$.*

Corollary 5 has been already shown in [17] in the case of $A = |T^*|$ and $B = |T|$ (i.e., T is log-hyponormal).

3. Karcher and power means of multi-variable matrices

In this section, we will discuss about only m–by–m matrices, hence $\mathcal{H}$ means $\mathbb{C}^m$. Before stating our discussion, we shall introduce some properties of power mean for the reader's convenience. Let $\omega = (w_1, ..., w_n) \in \Delta_n$ and $A_1, ..., A_n \in \mathbb{P}_m$. By the definition of power mean (1.2), we have

$$P_1(\omega; A_1, ..., A_n) = \sum_{i=1}^n w_iA_i \quad \text{and} \quad P_t(\omega; A_1, ..., A_n) = P_{-t}(\omega; A_1^{-1}, ..., A_n^{-1})^{-1}$$

for $t \in (0,1]$; especially

$$P_{-1}(\omega; A_1, ..., A_n) = \left(\sum_{i=1}^n w_iA_i^{-1}\right)^{-1}.$$

Moreover, we have

Lemma 6 ([8, 10, 11]). *The power mean $P_t(\omega; A_1, ..., A_n)$ is increasing for $t \in [-1, 1] \setminus \{0\}$, and*

$$\lim_{t \to 0} P_t(\omega; A_1, ..., A_n) = \Lambda(\omega; A_1, ..., A_n).$$

Henceforth, we use the symbol $P_0(\omega; A_1, ..., A_n)$ instead of $\Lambda(\omega; A_1, ..., A_n)$.

Theorem 7. *Let $A_1, ..., A_n$ be Hermitian matrices, and $\omega = (w_1, ..., w_n) \in \Delta_n$. Let $f(t)$ be a non-constant operator monotone function on $(0, \infty)$ with $f(1) = 1$. Then the following are equivalent:*

(i) $\sum_{i=1}^{n} w_i A_i \leq 0$,

(ii) $P_1(\omega; f(\lambda A_1 + I), ..., f(\lambda A_n + I)) = \sum_{i=1}^{n} w_i f(\lambda A_i + I) \leq I$ *for all sufficiently small* $\lambda \geq 0$,

(iii) *for each* $t \in [-1, 1]$, $P_t(\omega; f(\lambda A_1 + I), ..., f(\lambda A_n + I)) \leq I$ *for all sufficiently small* $\lambda \geq 0$.

Proof. Proof of (i) $\longrightarrow$ (ii). It is obvious that (i) implies $\sum_{i=1}^{n} w_i(\lambda A_i + I) \leq I$ for all $\lambda \geq 0$. Since $f(t)$ is an operator concave function with $f(1) = 1$, we have

$$I = f(I) \geq f\left(\sum_{i=1}^{n} w_i(\lambda A_i + I)\right) \geq \sum_{i=1}^{n} w_i f(\lambda A_i + I).$$

(ii) $\longrightarrow$ (iii) is given by only using Lemma 6, that is,

$$\begin{aligned} P_t(\omega; f(\lambda A_1 + I), ..., f(\lambda A_n + I)) &\leq P_1(\omega; f(\lambda A_1 + I), ..., f(\lambda A_n + I)) \\ &= \sum_{i=1}^{n} w_i f(\lambda A_i + I) \leq I. \end{aligned}$$

We shall prove (iii) $\longrightarrow$ (i). By Lemma 6, we have

$$\left(\sum_{i=1}^{n} w_i f(\lambda A_i + I)^{-1}\right)^{-1} \leq P_t(\omega; f(\lambda A_1 + I), ..., f(\lambda A_n + I)) \leq I.$$

Then we have

$$I \leq \sum_{i=1}^{n} w_i f(\lambda A_i + I)^{-1} \leq \left(\sum_{i=1}^{n} w_i f(\lambda A_i + I)^{-\frac{p}{\lambda}}\right)^{\frac{\lambda}{p}}$$

for $0 < \lambda \leq p$. Hence we have

$$I \leq \left(\sum_{i=1}^{n} w_i f(\lambda A_i + I)^{-\frac{p}{\lambda}}\right)^{\frac{1}{p}}.$$

By (2.2), we have

$$I \leq \left(\sum_{i=1}^{n} w_i e^{-pf'(1)A_i}\right)^{\frac{1}{p}} \quad \text{as } \lambda \to 0.$$

By Lemma 2, we have

$$I \leq \exp\left(\sum_{i=1}^{n} w_i \log e^{-f'(1)A_i}\right),$$

that is, (i). □

We especially consider the probability vector $\omega = (\frac{1}{n}, ..., \frac{1}{n})$ to obtain a multi-variable case of Theorem 3.

Theorem 8. *Let $A_1, ..., A_n$ be Hermitian matrices, and let f be a non-constant operator monotone function on $(0, \infty)$ with $f(1) = 1$. Then the following are equivalent:*

(i) $\sum_{i=1}^{n} A_i \leq 0$,

(ii) $\|x\|^n \leq \prod_{i=1}^{n} \|f(\lambda A_i + I)^{\frac{-1}{2}} x\|$ *for all sufficiently small $\lambda \geq 0$ and all $x \in \mathcal{H}$,*

(iii) $\|x\|^n \leq \prod_{i=1}^{n} \|e^{-pA_i} x\|$ *for all $x \in \mathcal{H}$ and all $p \geq 0$.*

Proof of Theorem 8. Assume (i). We have

$$\Lambda(f(\lambda A_1 + I), ..., f(\lambda A_n + I)) \leq I$$

for all sufficiently small $\lambda \geq 0$ by Lemma 6 and Theorem 7. Let $t_1, ..., t_n$ be positive numbers satisfying $\prod_{i=1}^{n} t_i = 1$. Using harmonic-geometric means inequality, we have

$$\begin{aligned} I &\geq \Lambda(f(\lambda A_1 + I), ..., f(\lambda A_n + I)) \\ &= \Lambda(t_1^{-1} f(\lambda A_1 + I), ..., t_n^{-1} f(\lambda A_n + I)) \geq \left(\sum_{i=1}^{n} \frac{t_i}{n} f(\lambda A_i + I)^{-1}\right)^{-1}, \end{aligned}$$

that is,

$$I \leq \sum_{i=1}^{n} \frac{t_i}{n} f(\lambda A_i + I)^{-1}.$$

Hence we have (ii) by Lemma 4. We next assume (ii). By Lemma 4, we have

$$I \leq \frac{1}{n} \sum_{i=1}^{n} t_i f(\lambda A_i + I)^{-1} \leq \left(\frac{1}{n} \sum_{i=1}^{n} t_i^{\frac{-p}{\lambda}} f(\lambda A_i + I)^{\frac{-p}{\lambda}}\right)^{\frac{\lambda}{p}}$$

for all $0 < \lambda \leq p$. Then

$$\frac{1}{n} \sum_{i=1}^{n} t_i^{\frac{-p}{\lambda}} f(\lambda A_i + I)^{\frac{-p}{\lambda}} \geq I,$$

and by Lemma 4, we obtain

$$\|x\|^n \leq \prod_{i=1}^{n} \|f(\lambda A_i + I)^{\frac{-p}{2\lambda}} x\|$$

holds for all $x \in \mathcal{H}$. Letting $\lambda \to 0$, we have

$$\|x\|^n \leq \prod_{i=1}^{n} \|e^{-\frac{pf'(1)}{2}A_i} x\|$$

holds for all $p > 0$ by (2.2). Replacing $pf'(1)/2$ into $p > 0$, we have (iii).

Lastly we assume (iii). By Lemma 4, we have

$$\frac{1}{n}\sum_{i=1}^{n} e^{-pA_i} \geq I,$$

and we obtain

$$\left(\frac{1}{n}\sum_{i=1}^{n} e^{-pA_i}\right)^{\frac{1}{p}} \geq I$$

for all $p > 0$. Hence by Lemma 2, we have (i). □

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